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THE ENERGY CONSERVATION EQUATION IN THE REFLECTOR MAPPING PROBLEM*

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Abstract—We consider here an inverse problem in geometric optics consisting of finding a reflector surface that transforms a given input energy density pattern into a prespecified output energy pattern on the far field. We derive a single explicit relation expressing the ray tracing and energy conservation equations in terms of the polar radius of the reflector and its first and second derivatives. Conditions are given for existence of a solution to a boundary value problem describing a reflector system which transforms a nonaxially symmetric input energy density into a uniform output energy distribution.

1. INTRODUCTION

In this paper, we use differential geometric methods to study the important practical problem of synthesis of single reflector antennas. Our main result here is a new form of the energy conservation equation relating the input and output energy densities. In contrast with other known forms of this equation (see [1–3]), the expression that we derive is in explicit, real valued form, and involves familiar geometric quantities such as the polar radius of the reflector and its second fundamental form. We analyze in our framework the synthesis problem as a boundary value problem for a second order nonlinear partial differential equation (PDE) of Monge-Ampere type and formulate sufficient conditions for its solvability. Previously, rigorous existence and uniqueness results were known only for axially symmetric reflector surfaces [4] and for a different formulation of the problem in which the input density was required to be uniform [5,6].

2. PRELIMINARIES

In three-dimensional space $\mathbb{R}^3$ let S be a unit sphere centered at some point O. Let Ω be a domain on S and $\bar{\Omega}$ its closure. Assume that $\bar{\Omega} \neq S$. Let (u^1, u^2) be some smooth local coordinates on the sphere S so that $\mathbf{m} = \mathbf{m}(u) \equiv \mathbf{m}(u^1, u^2)$ is the position vector of S. We assume that the coordinates u^1, u^2 are chosen so that $\langle \mathbf{m}, \mathbf{m}_1 \times \mathbf{m}_2 \rangle > 0$ in $\bar{\Omega}$; here $\mathbf{m}_i = \dfrac{\partial \mathbf{m}}{\partial u^i}$, $i = 1, 2$, and $\langle , \rangle$ denotes the scalar product in $\mathbb{R}^3$. As usual, we put $f(\mathbf{m}(u)) \equiv f(u^1, u^2)$ for any function $f : S \to \mathbb{R}$.

The first fundamental form $e = e_{ij}\, du^i du^j$ of S has coefficients $e_{ij} = \langle \mathbf{m}_i, \mathbf{m}_j \rangle$. Here, and everywhere below, the summation convention over repeated lower and upper indices is in effect, and we use the range of indices $1 \leq i, j, k, \ldots \leq 2$. The matrix $[e_{ij}]$ is symmetric and invertible; its inverse is denoted by $[e^{ij}]$.

Let ρ be a positive function in $\bar{\Omega}$ and set $\mathbf{r}(\mathbf{m}) = \rho(\mathbf{m})\mathbf{m}$. Then $\mathbf{r}$ defines a surface F projecting radially from O univalently onto $\bar{\Omega}$. The first fundamental form $g = g_{ij}\, du^i\, du^j$ of the surface F

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has coefficients $g_{ij} = \langle \mathbf{r}_i, \mathbf{r}_j \rangle = \rho_i \rho_j + \rho^2 e_{ij}$ where $\mathbf{r}_i = \dfrac{\partial \mathbf{r}}{\partial u^i}$ and $\rho_i = \dfrac{\partial \rho}{\partial u^i}$. The inverse of (g_{ij}) is given by

$$g^{ij} = \frac{1}{\rho^2}\left[e^{ij} - \frac{\rho_k e^{ki} \rho_l e^{lj}}{\rho^2 + |\tilde{\nabla}\rho|^2}\right], \tag{1}$$

where $|\tilde{\nabla} p|^2 = \rho_i \rho_j e^{ij}$.

Denote by $\mathbf{n}$ the unit normal vector field on F and assume that F is oriented so that $\langle \mathbf{m}, \mathbf{n} \rangle > 0$ everywhere on F. Then, clearly, $\langle \mathbf{r}, \mathbf{n} \rangle = \rho \langle \mathbf{m}, \mathbf{n} \rangle > 0$ on F. The vectors $\mathbf{m}_1(u)$, $\mathbf{m}_2(u)$, and $\mathbf{m}(u)$ form a basis of $\mathbb{R}^3$ at every $u \in \bar{\Omega}$, and $\mathbf{n}(u)$ can be expressed in terms of this "moving" basis as

$$\mathbf{n} = (\rho^2 + |\tilde{\nabla}\rho|^2)^{-1/2}(\rho \mathbf{m} - \tilde{\nabla}\rho), \tag{2}$$

where $\tilde{\nabla}\rho = \rho_i e^{ij} \mathbf{m}_j$.

Suppose a light ray originates at O in the direction $\mathbf{m}$ and is reflected off the surface F at the point $\mathbf{r}(\mathbf{m})$ in the direction $\mathbf{y}$. The surface F is called the *reflector*. By Snell's law,

$$\mathbf{y} = \mathbf{m} - 2\langle \mathbf{m}, \mathbf{n} \rangle \mathbf{n}, \tag{3}$$

and we may consider $\mathbf{y}$ as a map from $\bar{\Omega}$ into S. Put $\bar{\omega} = \mathbf{y}(\bar{\Omega})$. The set $\bar{\omega}$ is called the *far field* and $\mathbf{y}$ is called the *reflector map*. The area elements in $\bar{\Omega}$ and in its image $\bar{\omega}$ are related by means of the Jacobian determinant J of the map $\mathbf{y}$. It is given by

$$J = \pm \frac{|d\sigma(\mathbf{y}(\mathbf{m}))|}{d\sigma(\mathbf{m})} = \pm \frac{\sqrt{\det \langle \mathbf{y}_i(\mathbf{m}), \mathbf{y}_j(\mathbf{m}) \rangle}}{\sqrt{\det \langle \mathbf{m}_i, \mathbf{m}_j \rangle}}, \tag{4}$$

where $d\sigma$ is the area element in Ω and $\mathbf{y}_i = \dfrac{\partial \mathbf{y}}{\partial u^i}$. We assign a $\pm$ sign to the Jacobian according to whether $\mathbf{y}$ preserves the orientation or reverses it.

Let $I(\mathbf{m})$ be the energy density of the source located at O and $L(\mathbf{y})$ the output energy density in the reflected direction $\mathbf{y}$. According to the differential form of the energy conservation law [1],

$$L(\mathbf{y}(\mathbf{m}))J(\mathbf{y}(\mathbf{m})) = \pm I(\mathbf{m}). \tag{5}$$

We need to transform this general relation into a differential equation for the polar radius ρ of the reflector surface F. It follows from (4) that this is equivalent to finding such an expression for the Jacobian J.

It will be useful to have several different expressions for the vector functions $\mathbf{m}(u)$ and $\mathbf{y}(u)$. A simple calculation (that we omit) allows us to write an expression for $\mathbf{m}(u)$ in terms of the basis $\mathbf{r}_1(u)$, $\mathbf{r}_2(u)$, and $\mathbf{n}(u)$:

$$\mathbf{m} = \rho_i g^{ij} \mathbf{r}_j + \sqrt{1 - |\nabla\rho|^2}\, \mathbf{n}, \tag{6}$$

where $\nabla\rho = \rho_i g^{ij} \mathbf{r}_j$. Using (1), one can see that $|\nabla\rho|^2 < 1$. Furthermore, by (2) and (6) we have

$$\langle \mathbf{m}, \mathbf{n} \rangle = \frac{\rho}{\sqrt{|\tilde{\nabla}\rho|^2 + \rho^2}} = \sqrt{1 - |\nabla\rho|^2}. \tag{7}$$

We may also express $\mathbf{y}$ in terms of ρ, $\mathbf{m}$, and their derivatives. By Snell's law and (2), we get

$$\mathbf{y} = \mathbf{m} - 2\frac{\rho(\rho \mathbf{m} - \tilde{\nabla}\rho)}{|\tilde{\nabla}\rho|^2 + \rho^2}. \tag{8}$$

Using Snell's law and (7), we may also express $\mathbf{y}$ without explicit use of $\mathbf{m}$:

$$\mathbf{y} = \rho_i g^{ij} \mathbf{r}_j - \sqrt{1 - |\nabla\rho|^2}\, \mathbf{n}. \tag{9}$$

3. COMPUTATION OF THE JACOBIAN J

It follows from (4) that we need to compute $\det\langle \mathbf{y}_i, \mathbf{y}_j\rangle$.

PROPOSITION 3.1. *Let F be a reflector surface as in Section 1. Put*

$$H_{ij} = \nabla_{ij}\rho + \sqrt{1-|\nabla\rho|^2}\, b_{ij}, \tag{10}$$

where $\nabla_{ij} = \dfrac{\partial^2}{\partial u^i \partial u^j} - \Gamma_{ij}^k \dfrac{\partial}{\partial u^k}$ denote the second covariant derivatives in the metric g, and Γ_{ij}^k are the Christoffel symbols of the second kind of the metric g. Then

$$\sqrt{\det\langle \mathbf{y}_i, \mathbf{y}_j\rangle} = \frac{|\det H_{ij}|}{\sqrt{\det g_{ij}}} \frac{1}{\sqrt{1-|\nabla\rho|^2}}. \tag{11}$$

PROOF. Put $\rho^s = g^{sk}\rho_k$. We begin by showing that

$$\langle \mathbf{y}_i, \mathbf{y}_j\rangle = H_{ik}H_{js}\left(g^{ks} + \frac{\rho^k\rho^s}{1-|\nabla\rho|^2}\right). \tag{12}$$

We differentiate (9) covariantly relative to the metric g and obtain

$$\nabla_i \mathbf{y} = \mathbf{y}_i = \nabla_{ik}\rho g^{ks}\mathbf{r}_s + \rho_k g^{ks}\nabla_{si}\mathbf{r} + \frac{\rho_s g^{sk}\nabla_{ki}\rho}{\sqrt{1-|\nabla\rho|^2}}\mathbf{n} - \sqrt{1-|\nabla\rho|^2}\,\mathbf{n}_i,$$

where ∇_i is the first covariant derivative relative to the metric g. To simplify this expression, we use the classical derivation formulas

$$\nabla_{ij}\mathbf{r} = b_{ij}\mathbf{n},\ \ \mathbf{n}_i = -\, b_{is}g^{sk}\mathbf{r}_k, \tag{13}$$

where the b_{ij}'s are the coefficients of the second fundamental form of the reflector surface F and $\mathbf{n}_i = \dfrac{\partial \mathbf{n}}{\partial u^i}$. Now we see that

$$\begin{aligned} \mathbf{y}_i &= \left[\nabla_{si}\rho + \sqrt{1-|\nabla\rho|^2}\, b_{si}\right] g^{sk}\mathbf{r}_k + \left[\nabla_{si}\rho + \sqrt{1-|\nabla\rho|^2}\, b_{si}\right] g^{sk}\frac{\rho_k\mathbf{n}}{\sqrt{1-|\nabla\rho|^2}} \\ &= H_{is}g^{sk}\left(\mathbf{r}_k + \frac{\rho_k}{\sqrt{1-|\nabla\rho|^2}}\mathbf{n}\right), \end{aligned} \tag{14}$$

and evaluating $\langle \mathbf{y}_i, \mathbf{y}_j\rangle$, we obtain (12). On the other hand, we have

$$\det\left(g^{ij} + \frac{\rho^i\rho^j}{1-|\nabla\rho|^2}\right) = \frac{1}{\det(g_{ij})(1-|\nabla\rho|^2)}. \tag{15}$$

Consequently, evaluating the determinants on both sides of (12), we obtain (11). The proposition is proved.

The following proposition gives a clear geometric description of H_{ij}.

PROPOSITION 3.2. *Let H_{ij}, b_{ij}, and e_{ij} be as before. Then*

$$H_{ij} = 2b_{ij}\langle \mathbf{m}, \mathbf{n}\rangle + \rho e_{ij}. \tag{16}$$

PROOF. It follows from (10) and (7) that

$$H_{ij} = b_{ij}\langle \mathbf{m}, \mathbf{n}\rangle + \nabla_{ij}\rho. \tag{17}$$

Since $\rho^2 = \langle \mathbf{r}, \mathbf{r}\rangle$, and $\rho\rho_i = \langle \mathbf{r}, \mathbf{r}_i\rangle$, we obtain with the use of (13)

$$\rho_i\rho_j + \rho\nabla_{ij}\rho = \langle \mathbf{r}_i, \mathbf{r}_j\rangle + \langle \mathbf{r}, \nabla_{ij}\mathbf{r}\rangle = g_{ij} + b_{ij}\langle \mathbf{r}, \mathbf{n}\rangle.$$

Because $g_{ij} = \rho_i\rho_j + \rho^2 e_{ij}$, we get

$$\rho\nabla_{ij}\rho = b_{ij}\langle \mathbf{r}, \mathbf{n}\rangle - \rho_i\rho_j + g_{ij} = b_{ij}\langle \mathbf{r}, \mathbf{n}\rangle + \rho^2 e_{ij} = \rho b_{ij}\langle \mathbf{m}, \mathbf{n}\rangle + \rho^2 e_{ij}.$$

Dividing by ρ and substituting in (17), we obtain (16). The proposition is proved.

Evaluating the determinant of g_{ij} in terms of ρ and its derivatives, we find that

$$\det(g_{ij}) = \rho^2(\rho^2 + |\tilde{\nabla}\rho|^2)\det(e_{ij}). \tag{18}$$

We use (16) and (18) to rewrite (11) as

$$\sqrt{\det\langle \mathbf{y}_i, \mathbf{y}_j\rangle} = \frac{1}{\rho^2}\frac{|\det(2b_{ij}\langle \mathbf{m}, \mathbf{n}\rangle + \rho e_{ij})|}{\sqrt{\det(e_{ij})}}. \tag{19}$$

It follows from (4) and (19) that

$$J = \pm\frac{\sqrt{\det\langle \mathbf{y}_i, \mathbf{y}_j\rangle}}{\sqrt{\det\langle \mathbf{m}_i, \mathbf{m}_j\rangle}} = \frac{1}{\rho^2}\frac{\det(2b_{ij}\langle \mathbf{m}, \mathbf{n}\rangle + \rho e_{ij})}{\det(e_{ij})} \equiv G(\rho). \tag{20}$$

The following expression for the second fundamental form in terms of ρ is derived in [7]:

$$b_{ij} = \frac{\rho\tilde{\nabla}_{ij}\rho - \rho^2 e_{ij} - 2\rho_i\rho_j}{\sqrt{\rho^2 + |\tilde{\nabla}\rho|^2}}, \tag{21}$$

where $\tilde{\nabla}_{ij} = \dfrac{\partial^2}{\partial u^i \partial u^j} - \tilde{\Gamma}^k_{ij}\dfrac{\partial}{\partial u^k}$ denote the second covariant derivatives in the metric e on S, and $\tilde{\Gamma}^k_{ij}$ are the Christoffel symbols of the second kind of the metric e. Combining expressions (21), (20), and (7), we get

$$G(\rho) = \frac{1}{(\rho^2 + |\tilde{\nabla}\rho|^2)^2}\frac{\det\{2\rho\tilde{\nabla}_{ij}\rho - (\rho^2 - |\tilde{\nabla}\rho|^2)e_{ij} - 4\rho_i\rho_j\}}{\det(e_{ij})}. \tag{22}$$

4. THE REFLECTOR PROBLEM

In the *reflector problem*, we are given the input domain $\bar{\Omega} \subset S$ and the (future) far field $\bar{\omega} \subset S$ as well as positive density functions $I : \bar{\Omega} \to (0, \infty)$ and $L : \bar{\omega} \to (0, \infty)$. We have to find a reflector surface F subject to the requirements:

(i) the rays originating at O and going through points of $\bar{\Omega}$ project F univalently onto $\bar{\Omega}$;
(ii) F is required to be such that the reflector map $\mathbf{y}$ is a diffeomorphism of $\bar{\Omega}$ onto the far field $\bar{\omega}$;
(iii) for the given input energy density $I(\mathbf{m})$ from the source O, the output energy density after reflection off F in the direction $\mathbf{y} \in \omega$ is equal to $L(\mathbf{y})$.

We combine equations (5) and (20) and obtain an analytic expression of condition (iii) as a PDE for the polar radius ρ of the unknown reflector F:

$$L(\mathbf{y}(\mathbf{m}))(G(\rho))(\mathbf{m}) = \pm I(\mathbf{m}), \qquad \mathbf{m} \in \Omega. \tag{23}$$

The "+" or "−" sign is taken, depending on whether we are looking for a reflector for which the map $\mathbf{y}$ preserves or reverses the orientation of Ω. Note that G is an elliptic (positive or negative) operator when the "+" is taken in equation (23) and that G is a hyperbolic operator when the "−" is taken.

In addition, if the map $\mathbf{y}$ satisfies (ii) then it is necessary that

$$\mathbf{y} : \partial\Omega \to \partial\omega. \tag{24}$$

In equation (8), $\mathbf{y}$ is expressed in terms of ρ and its derivatives and therefore (24) is a condition on ρ and its derivatives on $\partial\Omega$. This gives us a boundary condition to be satisfied by solutions of (23). It expresses the *requirement that the boundary of Ω be mapped onto the boundary of ω.*

Integrating (5) and using the change of variables formula, we obtain a necessary condition on the data for the problem to have a solution:

$$\int_\omega L(\mathbf{y})\,d\sigma(\mathbf{y}) = \int_\Omega I(\mathbf{m})\,d\sigma(\mathbf{m}), \tag{25}$$

where $d\sigma(\mathbf{y})$ is the area element in ω.

Now we formulate conditions for existence and uniqueness of solutions to (23), (24). We are considering only the case of the "+" sign in (23). Suppose that the input domain Ω and far field ω are axially symmetric and co-axial about the z axis. Using spherical coordinates (α, β), where $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ and $0 \leq \beta \leq 2\pi$, we describe Ω and ω as follows:

$$\bar{\Omega} = \left\{(\alpha, \beta) \mid \bar{\alpha} \leq \alpha \leq \frac{\pi}{2}\right\}, \bar{\alpha} \in \left(0, \frac{\pi}{2}\right); \quad \bar{\omega} = \left\{(\alpha, \beta) \mid -\frac{\pi}{2} \leq \alpha \leq \tilde{\alpha}\right\}, \tilde{\alpha} \in \left(-\frac{\pi}{2}, 0\right).$$

In this setting, the boundary condition (24) takes the explicit form (see [4, Section 2.3])

$$\langle \mathbf{y}(\bar{\alpha}, \beta), (0, 0, -1)\rangle = -\sin \tilde{\alpha}. \tag{26}$$

Let L_0 be a given positive constant and

$$H = \left\{I \in C^{0,\delta}(\bar{\Omega}) \mid \int_{\Omega} I d\sigma = 2\pi L_0(1 + \sin \tilde{\alpha})\right\},$$

where $\delta \in (0, 1)$. Note that H describes all the functions in $C^{0,\delta}(\bar{\Omega})$ which satisfy the integral form of the conservation of energy requirement (25) with $L = L_0$.

THEOREM 4.1. *Let $\bar{I}$ be a positive function of class $C^1\left[0, \frac{\pi}{2}\right]$, and Ω, ω, and L_0 be such that*

$$\int_{\bar{\alpha}}^{\frac{\pi}{2}} \bar{I}(\tau) \cos \tau d\tau = L_0(1 + \sin \tilde{\alpha}). \tag{27}$$

Assume further that the function $\bar{I}$ is extended to the entire $\bar{\Omega}$ by setting $\bar{I}(\alpha, \beta) \equiv \bar{I}(\alpha)$ for any $(\alpha, \beta) \in \left[\bar{\alpha}, \frac{\pi}{2}\right] \times [0, 2\pi]$. Then there exists an $\epsilon > 0$ such that for any $I \in H$ satisfying $||I - \bar{I}|| < \epsilon$, where $||\,||$ denotes the Holder norm in $C^{0,\delta}(\bar{\Omega})$, the equation (23) with $L(\mathbf{y}) \equiv L_0$ and "+" sign in the right hand side with boundary condition (24) admits two classes of positive solutions $\rho \in C^{2,\delta}(\bar{\Omega})$, determined uniquely within each class up to a positive multiplicative constant. Furthermore, the corresponding reflector surfaces given by $\mathbf{r}(\mathbf{m}) = \rho(\mathbf{m})\mathbf{m}$ satisfy the conditions (i)–(iii) *as described in the beginning of this section.*

The role of the function $\bar{I}$ is that we first construct an axially symmetric solution of (23), (24) with $\bar{I}$ as the right hand side in (23). The nonaxially symmetric solutions are constructed as perturbations of the axially symmetric ones. However, the actual proof of the theorem is obtained by first reducing the boundary value problem (23), (24) to a boundary value problem for which existence and uniqueness results have been established in [6]. This reduction is based on the observation that upon setting $\rho = 1/p$ in the equation (22) for G and in the boundary condition (24), we obtain a new equation and boundary condition identical with the equation and boundary condition studied in [6]. Once it is shown that the problem (23), (24) has a solution, one uses some additional topological arguments to show existence of a reflecting surface satisfying the requirements (i)–(iii).

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